Research plan:

Sub-Riemannian congested optimal transport and associated quasilinear PDEs

Scope of this project is to develop two of the objectives of the project PRIN2022 Regularity problems in sub-Riemannian structures

- Properties of sub-Riemannian congested optimal transport
- Interior regularity of the associated p-Laplace type equation

The problem of congested optimal transportation has been introduced in the Euclidean setting in [3]. It is a modification of the classical Monge Kantorovich problem, which moves probability measures along admissible paths. It is based on the notion of traffic intensity which is an estimate of congestion in the network. The optimal configuration, called Wardrop equilibrium, is a measure, minimizing a suitable optimization problem. [2] and [3] introduced equivalent formulations of the problem in terms of minima of convex functionals, deeply simplifying the problem of finding a solution. We are interested in the analogous results in the Heisenberg setting with a sub-Riemannian metric. Optimal transportation theory of Monge-Kantorovich type in the Heisenberg Group was introduced in [1], and developed in the full generality of sub-Riemannian spaces in [5].

The problem of congested optimal transportation was recently introduced in the Heisenberg group with a sub-Riemannian metric. In this case the metric is defined only on a distribution, sub-bundle of the tangent bundle, and propagation takes place along curves tangent to the distribution at every point, also called horizontal curves. Hence, while defining the congested transportation in this setting, only horizontal curves are considered, leading to significant technical difficulties in the extension of the theory. However, a characterization of congested traffic a characterization analogous of one proposed in [3] is already known in the Heisenberg group.

Here we plan to apply the duality procedure and express the problem in terms of a p-Laplacian-type sub-Riemannian equation, extending to this context the results in [2]. The problem will be studied in the Heisenberg to begin with, but it could be also be interesting to pose the problem in more general sub-Riemannian structures, starting from H-type structures.

Then we will study the regularity of the resulting quasilinear problem. In the Euclidean setting, it is expressed as a degenerate quasilinear second order PDE. The principal part of the operator has a p-Laplacian type growth (homogeneous in all directions or orthotropic) when the modulus of the gradient is sufficiently big, so that Lipschitz regularity of the gradient was established in [2], while it vanishes identically when the modulus of the gradient is smaller than one, so that higher regularity is not expected. The result is much more difficult in the degenerate Heisenberg setting, since the equation will be expressed in terms of vector fields which do not commute. However, the regularity for the standard p-Laplacian operator has been established by [6], and [4]. On the other side, the regularity of its orthotropic counterpart is not known. We will start from studying the regularity of solutions of this equation. We will introduce a technique based on Caccioppoli-type inequalities. When formally computing the equation satisfied by the horizontal first order derivatives, commutators will show up. For this reason, we shall start by studying the regularity of the derivative in the direction of the commutator. Ad hoc Caccioppoli-type inequalities for products of derivatives of solutions and their commutators will be probably needed. Then we will try to establish the estimate of the gradient even when the fundamental form vanishes for small values of the gradient.

The ideal candidate should have experience in one or more topics of the project: the optimal mass transportation, the regularity of p-Laplacian type equation, preferably in the sub-Riemannian setting.

Activity plan:

The activity will take place at the Department of Mathematics, University of Bologna, within the funded project PRIN 2022F4F2LH "Regularity problems in sub-Riemannian structures".Coord. CITTI Giovanna, ERC: PE1_8, CUP J53D23003760006.. The researcher will also visit prof. Brasco (U. Ferrara) and prof. X. Zhong (U. Helsinki) to discuss different aspects of the project.

He will also present his results to conferences and seminars.

References

[1] L. Ambrosio, S. Rigot, Optimal mass transportation in the Heisenberg group. J. Funct. Anal. 208, (2004) 261–301.

[2] L. Brasco, G. Carlier, and F. Santambrogio. Congested traffic dynamics, weak flows and very degenerate elliptic equations. In: J. Math. Pures Appl. (9) 93.6 (2010), 652–671

[3] G. Carlier, C. Jimenez, and F. Santambrogio. Optimal transportation with traffic congestion and Wardrop equilibria. SIAM J. Control Optim. 47.3 (2008), 1330–1350.

[4] G. Citti, S. Mukherjee, Regularity of quasi-linear equations with Hörmander vector fields of step two, Adv. Math., 408, 2022.

[5] A. Figalli L. Rifford, Mass transportation on sub-Riemannian manifolds, Geom. Funct. Anal. 20 (2010), 124–59

[6] X. Zhong, Regularity for variational problems in the Heisenberg group. preprint https://arxiv.org/abs/1711.03284